

# **Worcester County Mathematics League**

**Varsity Meet 1  
October 3, 2012**

<p><b>COACHES' COPY ROUNDS, ANSWERS, AND SOLUTIONS</b></p>
----------------------------------------------------------------





Varsity Meet 1 – October 3, 2012  
Round 1: Arithmetic

All answers must be in simplest exact form in the answer section

**NO CALCULATOR ALLOWED**

1. Evaluate:  $2013^2 + 2012^2 - 2(2012)(2013)$ .

2. Let the operation  $x \spadesuit y$  be defined as  $\frac{x+y}{xy}$ . Given that  $a \spadesuit (2 \spadesuit 3) = 4 \spadesuit 1$ , find the value of  $a$ .

3. Evaluate the following:

$$-|-3|^2 + (-1)^{-12} \div (\sqrt{1} + 1) \cdot 2^2 + \frac{6(2+1)}{\sqrt[3]{27}}$$

**ANSWERS**

(1 pt.) 1. \_\_\_\_\_

(2 pts.) 2. \_\_\_\_\_

(3 pts.) 3. \_\_\_\_\_





Varsity Meet 1 – October 3, 2012  
Round 2: Algebra I

All answers must be in simplest exact form in the answer section

**NO CALCULATOR ALLOWED**

1. Three measurements are recorded in an experiment. Two are measured as 64 and 78. The average is 80. What is the third measurement?

2. Simplify:

$$\frac{\frac{p}{q} - \frac{p-q}{p+q}}{\frac{q}{p} + \frac{p-q}{p+q}}$$

3. Given that  $x + \frac{1}{x} = 5$ , find  $x^3 + \frac{1}{x^3}$ .

**ANSWERS**

(1 pt.) 1. \_\_\_\_\_

(2 pts.) 2. \_\_\_\_\_

(3 pts.) 3. \_\_\_\_\_





Varsity Meet 1 – October 3, 2012  
Round 3: Set Theory

*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

1. Let  $A = \{\text{multiples of 3 between 1 and 100}\}$  and let  $B = \{\text{multiples of 7 between 1 and 100}\}$ . How many elements are in the intersection  $A \cap B$ ?

2. Of the 400 students at Hogwarts, 150 report that they are comfortable with Transfiguration, 200 say they are comfortable with Potions, and 80 say they are not comfortable with either subject. Based on this information, how many students feel comfortable with both subjects?

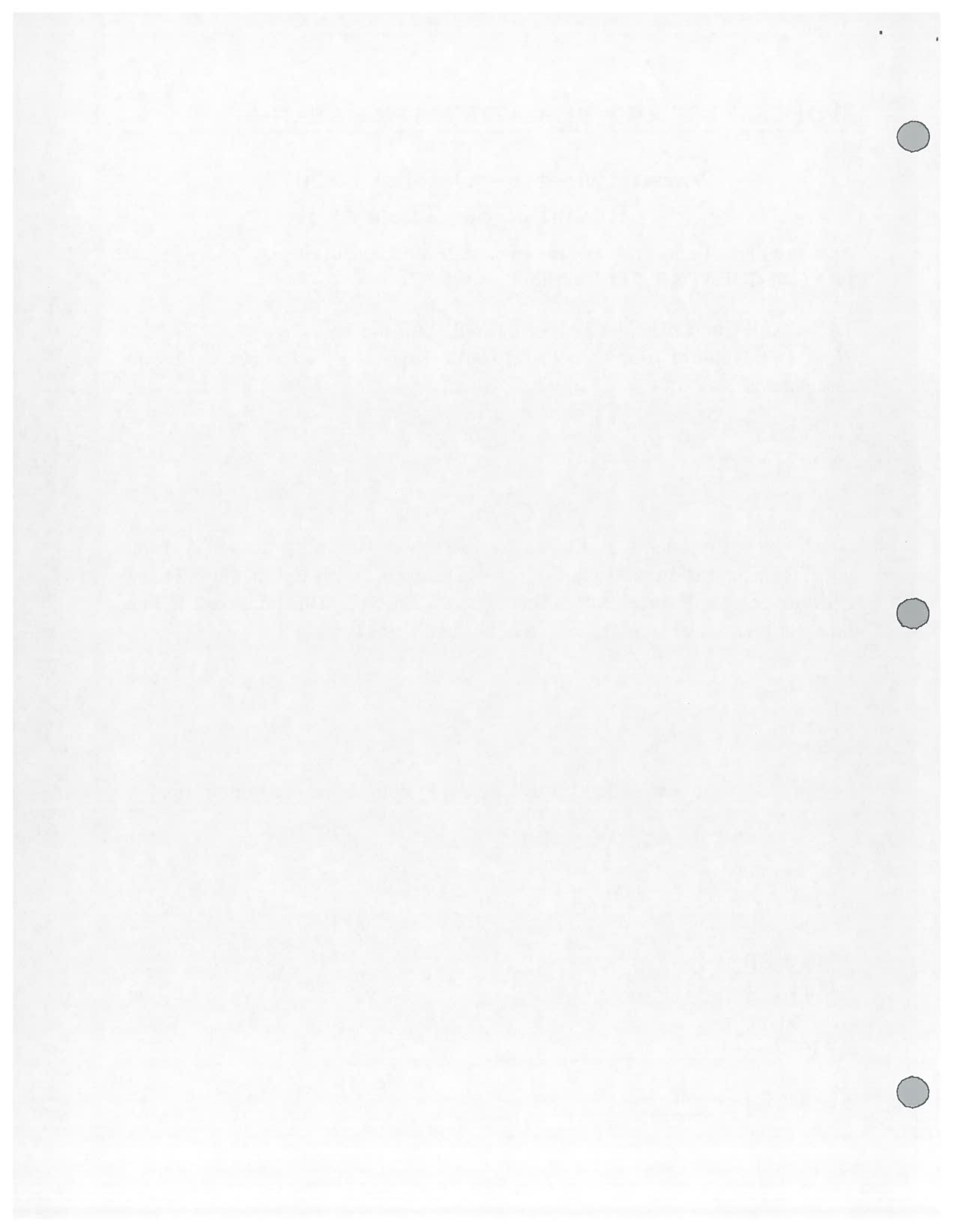
3. How many subsets of  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  contain at least one prime?

**ANSWERS**

(1 pt.) 1. \_\_\_\_\_

(2 pts.) 2. \_\_\_\_\_

(3 pts.) 3. \_\_\_\_\_







Varsity Meet 1 – October 3, 2012

Round 4: Measurement

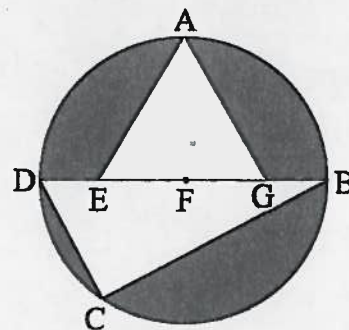
All answers must be in simplest exact form in the answer section

**NO CALCULATOR ALLOWED**

1. Find the area of the largest triangle that can be inscribed in a semicircle with radius 2.

2. The radius of a cylinder is 14 units and the height is 5 units. A positive number can be added to either the radius or the height to give the same increase in volume. What is this number?

3. In circle  $F$  with radius  $r$ ,  $DC = r$  and  $\triangle AEG$  is equilateral with  $\overline{AF}$  as an altitude. If the area of the shaded region can be written as  $\frac{P\pi r^2 + Qr^2\sqrt{3}}{6}$ , find the value of  $P + Q$ .



**ANSWERS**

(1 pt.) 1. \_\_\_\_\_

(2 pts.) 2. \_\_\_\_\_

(3 pts.) 3. \_\_\_\_\_

MEMORANDUM FOR THE DIRECTOR

DATE: 12/15/2001

TO: DIRECTOR, FBI

FROM: SAC, [Redacted]

SUBJECT: [Redacted]

1. On 12/14/2001, [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

2. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

3. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

4. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

5. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

6. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

7. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

8. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

9. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

10. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

11. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

12. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

13. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

14. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

15. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

16. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

17. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.

18. [Redacted] advised that [Redacted] had been observed at [Redacted] on 12/14/2001.



**Varsity Meet 1 – October 3, 2012**  
**Round 5: Polynomial Equations**

*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

1. When written in standard form, what is the sum of the coefficients of the quintic polynomial  $(13x - 15)^5$ ?
  
  
  
  
  
  
  
  
  
  
2. One root of  $5x^2 + 34x = k$  is  $-8$ . Find the other root.
  
  
  
  
  
  
  
  
  
  
3. If the roots of  $5x^2 - 13x + 2012 = 0$  are  $m$  and  $n$ , find the quadratic polynomial with roots  $m-3$  and  $n-3$  and write it in the form  $ax^2 + bx + c = 0$  with  $a, b, c$  relatively prime integers and  $a > 0$ .

**ANSWERS**

(1 pt.) 1. \_\_\_\_\_

(2 pts.) 2. \_\_\_\_\_

(3 pts.) 3. \_\_\_\_\_

THE UNIVERSITY OF CHICAGO

PHYSICS DEPARTMENT

5720 S. UNIVERSITY AVE.

CHICAGO, ILL. 60637

TEL: 773-936-3700

FAX: 773-936-3700

WWW.PHYSICS.UCHICAGO.EDU

PHYSICS 435

LECTURE 1

1.1. THE CLASSICAL LIMIT

1.2. QUANTUM MECHANICS

1.3. THE SCHROEDINGER EQUATION

1.4. THE HEISENBERG PICTURE

1.5. THE DIRAC EQUATION

1.6. SUMMARY



Varsity Meet 1 – October 3, 2012  
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

**APPROVED CALCULATORS ALLOWED**

1. Let  $x \# y = x^2 - 2xy + y^2$  and let  $x \# y = x^2 + 2xy + y^2$ . Find the value of

$$\frac{(a \# b)b(d \# c)}{bbd}$$

if  $a = 3$ ,  $b = -2$ ,  $c = -1$ , and  $d = 2$ .

2. A driver averaged 42 mph traveling to work. At what average speed must she return the same distance in order to average 48 mph for the round trip?
3. The universal set for this problem is  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Given that  $A = \{1, 2, 3, 4\}$ ,  $B = \{2, 4, 6, 8\}$ , and  $S'$  denotes the complement of set  $S$ , find

$$[(A' \cup B') \cap (A \cup B)] \cup A'$$

4. A right circular cone with a base of radius 1 is inscribed in a sphere of radius 2 such that its volume is maximal. What is the ratio of the volume of the cone to the volume of the sphere? Express your answer as a fraction.
5. Find all possible values of  $x$  that satisfy the equation

$$-11x + 6x\sqrt{x+1} + 9\sqrt{x+1} = 11.$$

6. Express the product  $(0.\overline{36}) \times (0.\overline{15})$  as a fraction in lowest terms.
7. Quadrilateral  $ABCD$  is inscribed in a circle. If  $m\angle A = (6x + 32)^\circ$ ,  $m\angle B = (10x + 5)^\circ$ , and  $m\angle C = (3x + 40)^\circ$ , find the degree-measure of angle  $D$ .
8. In a race, if Christina's running speed was  $\frac{3}{4}$  of Angela's and Moira's speed was  $\frac{2}{3}$  of Christina's, then Angela's speed was how many times the average of the other two runners' speeds?
9. Let  $S$  be the set of all positive integers that leave a remainder of 4 when divided by 7 and a remainder of 2 when divided by 11. What is the smallest element of  $S$ ?





**Varsity Meet 1 – October 3, 2012  
TEAM ROUND ANSWER SHEET**

*All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)*

1. \_\_\_\_\_

2. \_\_\_\_\_ mph

3. { \_\_\_\_\_ }

4. \_\_\_\_\_

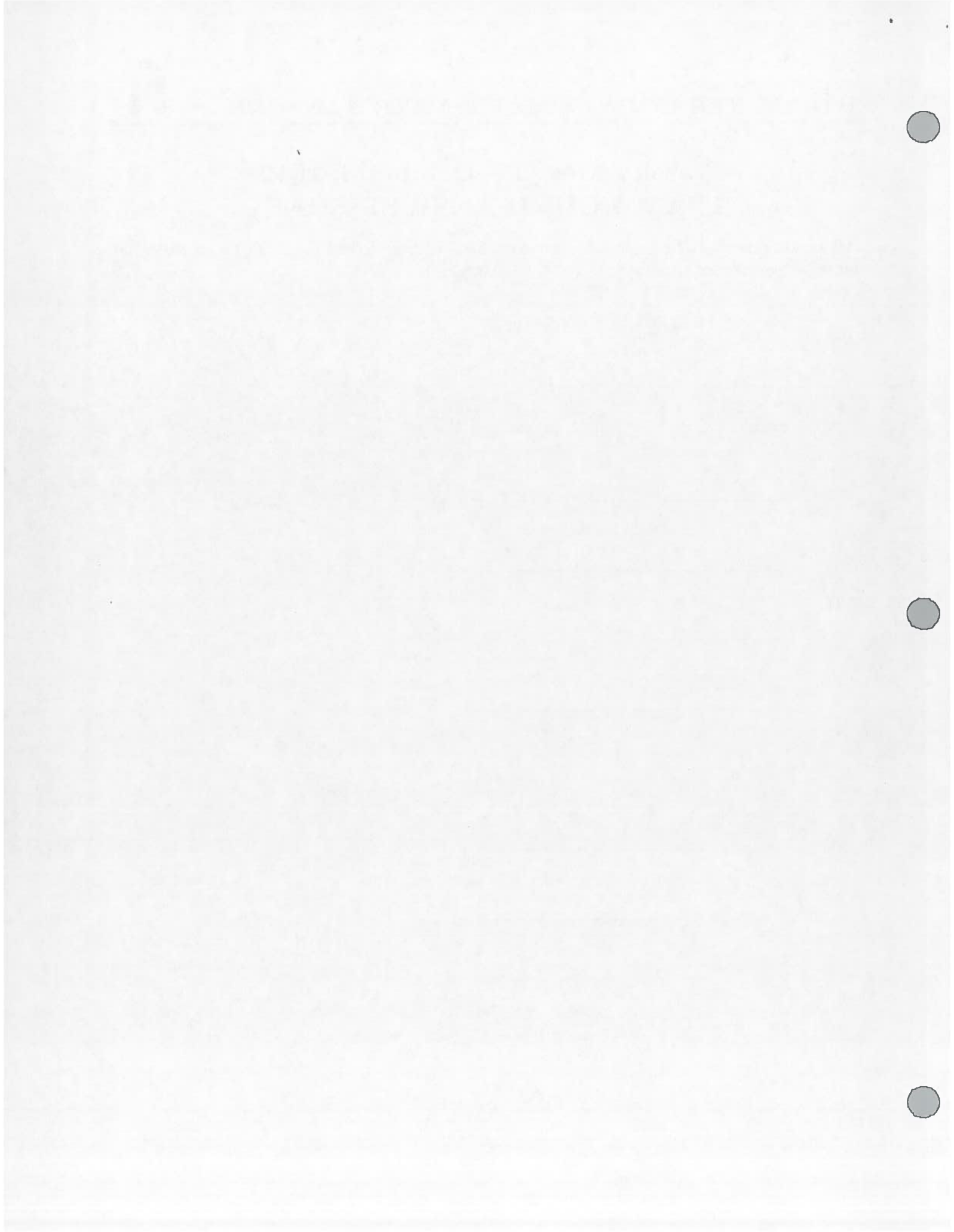
5. \_\_\_\_\_

6. \_\_\_\_\_

7. \_\_\_\_\_ °

8. \_\_\_\_\_

9. \_\_\_\_\_







Varsity Meet 1 – October 3, 2012  
ANSWERS

**ROUND 1**

(Leicester, Doherty, Tahanto)

1. 1
2. 20
3. -1

**ROUND 2**

(Bromfield, Notre Dame, QSC)

1. 98
2.  $p/q$
3. 110

**ROUND 3**

(Shepherd Hill, Hudson, QSC)

1. 4
2. 30
3. 480

**ROUND 4**

(Notre Dame, Hudson, Worc Academy)

1. 4
2.  $56/5 = 11\frac{1}{5} = 11.2$
3. 1

**ROUND 5**

(QSC, Auburn, QSC)

1. -32
2.  $6/5 = 1\frac{1}{5} = 1.2$
3.  $5x^2 + 17x + 2018 = 0$  (need the = 0)

**TEAM ROUND**

(Shepherd Hill, Worcester Academy, Quaboag, Algonquin, QSC, Quaboag, Hudson, Hudson, QSC)

1. 0
2. 56
3. {5, 7, 9} (in any order)
4.  $\frac{2 + \sqrt{3}}{32} \approx 0.117$
5. -1, -8/9, 5/4 (need all 3, in any order)
6. 1/18 (must be a fraction)
7. 55°
8.  $8/5 = 1\frac{3}{5} = 1.6$
9. 46





Varsity Meet 1 – October 3, 2012  
 FULL SOLUTIONS

ROUND 1

- From the identity  $(x - y)^2 = x^2 - 2xy + y^2$ , we have that  $2013^2 + 2012^2 - 2(2012)(2013) = (2013 - 2012)^2 = \boxed{1}$ .
- First, we have  $2\spadesuit 3 = 5/6$  and  $4\spadesuit 1 = 5/4$ , so we must solve the equation  $\frac{a + 5/6}{5a/6} = \frac{5}{4}$ .  
 Cross-multiplying, we have  $a = \boxed{20}$ .
- Following order of operations,

$$\begin{aligned} -|-3|^2 + (-1)^{-12} \div (\sqrt{1} + 1) \cdot 2^2 + \frac{6(2+1)}{\sqrt[3]{27}} &= -9 + 1 \div 2 \times 4 + 6 \\ &= -9 + \frac{1}{2} \times 4 + 6 \\ &= -9 + 2 + 6 \\ &= \boxed{-1}. \end{aligned}$$

In particular, note that division and multiplication are on the same level and are evaluated from left to right, so the answer is NOT  $-9 + \frac{1}{8} + 6 = -23/8$ .

ROUND 2

- The average is 80, so the sum is  $3(80) = 240$ . Therefore the third measurement is  $240 - 64 - 78 = \boxed{98}$ .
- Find a common denominator for both the numerator and denominator of the large fraction:

$$\frac{\frac{p^2 + pq - pq + q^2}{q(p + q)}}{\frac{pq + q^2 + p^2 - pq}{p(p + q)}}$$

Cancel out the common factors of  $(p + q)$  in both denominators and cancel out the  $+pq$  and  $-pq$  in each numerator:

1. The first part of the document is a list of names and addresses of the members of the committee.

2. The second part of the document is a list of names and addresses of the members of the committee.

3. The third part of the document is a list of names and addresses of the members of the committee.

4. The fourth part of the document is a list of names and addresses of the members of the committee.

5. The fifth part of the document is a list of names and addresses of the members of the committee.

6. The sixth part of the document is a list of names and addresses of the members of the committee.

7. The seventh part of the document is a list of names and addresses of the members of the committee.

8. The eighth part of the document is a list of names and addresses of the members of the committee.

9. The ninth part of the document is a list of names and addresses of the members of the committee.

10. The tenth part of the document is a list of names and addresses of the members of the committee.

11. The eleventh part of the document is a list of names and addresses of the members of the committee.

12. The twelfth part of the document is a list of names and addresses of the members of the committee.



$$\frac{\frac{p^2 + q^2}{q}}{p}$$

This is equal to  $\boxed{p/q}$ .

3. Recall the identity  $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ . Let  $a = x$  and  $b = 1/x$  so we have  $a + b = 5$  and  $ab = 1$ .

$$\text{Then, } x^3 + \frac{1}{x^3} = a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 5^3 - 3(1)(5) = 125 - 15 = \boxed{110}.$$

### ROUND 3

1. Since 3 and 7 are relatively prime, only multiples of 21 are in the intersection. There are  $\boxed{4}$  multiples of 21 between 1 and 100.
2. There are  $400 - 80 = 320$  students comfortable with at least one subject. From inclusion-exclusion, the number of students that feel comfortable with both subjects is  $150 + 200 - 320 = \boxed{30}$ .
3. There are a total of  $2^9 = 512$  subsets. The primes are 2, 3, 5, 7 so five elements are not primes (1 is a unit, neither prime nor composite). Therefore there are  $2^9 - 2^5 = 512 - 32 = \boxed{480}$  subsets that contain at least one prime.

### ROUND 4

1. The base is a diameter of the semicircle, so it has length 4. The area of a triangle is one-half of the base multiplied by the height. The height is maximized when it equals the radius (2). Hence the area is  $\frac{1}{2}(4)(2) = \boxed{4}$ .
2. We have

$$\begin{aligned} 5(14 + x)^2 &= 14^2(5 + x) \\ 5(196 + 28x + x^2) &= 196(5 + x) \\ 140x + 5x^2 &= 196x \\ 5x^2 - 56x &= 0 \end{aligned}$$

So the answer is  $\boxed{56/5}$ .

1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes the need for transparency and accountability in financial reporting.

2. The second part of the document outlines the various methods and techniques used to collect and analyze data. It covers both qualitative and quantitative research approaches, highlighting their strengths and limitations.

3. The third part of the document focuses on the ethical considerations surrounding data collection and analysis. It discusses the importance of informed consent, confidentiality, and the responsible use of research findings.

4. The fourth part of the document addresses the challenges and limitations of data analysis. It explores issues such as data quality, missing data, and the potential for bias in statistical inference.

5. The fifth part of the document discusses the application of data analysis in various fields, including business, social sciences, and healthcare. It provides examples of how data-driven insights can inform decision-making and improve outcomes.

6. The sixth part of the document concludes by summarizing the key findings and implications of the research. It emphasizes the need for continued research and innovation in data analysis to address emerging challenges and opportunities.

7. The seventh part of the document provides a list of references and sources used in the research. It includes books, articles, and online resources that provide further information on the topics discussed in the document.

8. The eighth part of the document contains a list of appendices and supplementary materials. These materials provide additional data, tables, and figures that support the findings and conclusions of the research.

9. The ninth part of the document includes a list of acknowledgments and a list of authors. It expresses gratitude to the individuals and organizations that provided support and assistance during the research process.

10. The tenth part of the document contains a list of contact information and a list of related works. It provides details on how to reach the authors and offers suggestions for further reading on the topic.



3. The area of the circle is  $\pi r^2$ , so  $P = 6$ . The area of the equilateral triangle is  $r^2/\sqrt{3} = \frac{r^2\sqrt{3}}{3}$ . The area of the right triangle (a 30-60-90 triangle based on the 1 : 2 ratio of side lengths  $DC : DB$ ) is  $\frac{r^2\sqrt{3}}{2}$ . Hence  $Q = -5$ . The answer is therefore  $P + Q = \boxed{1}$ .

## ROUND 5

- The sum of the coefficients of any polynomial  $p$  is equal to  $p(1)$ . Therefore, the sum of the coefficients of  $(13x - 15)^5$  is  $(13 \cdot 1 - 15)^5 = (-2)^5 = \boxed{-32}$ .
- METHOD I:** Since  $k$  is a constant, the sum of the roots of the polynomial  $5x^2 + 34x - k = 0$  is  $-34/5$ . We are given that one root is  $-8$ , so the other root must be  $-34/5 - (-8) = \boxed{6/5}$ .  
**METHOD II:** Plug in the root  $-8$  to find  $k = 5(-8)^2 + 34(5) = 320 - 272 = 48$ . Therefore, the polynomial is  $5x^2 + 34x - 48 = 0$  which can be factored as  $(x+8)(5x-6) = 0$ . Hence the other root is  $\boxed{6/5}$ , as before.
- Each root of the new polynomial is 3 less than that of the given polynomial, so this is a change of base from  $x$  to  $x + 3$ . Our desired polynomial is therefore

$$\begin{aligned} 5(x+3)^2 - 13(x+3) + 2012 &= 0 \\ 5(x^2 + 6x + 9) - 13x - 39 + 2012 &= 0 \\ 5x^2 + 30x + 45 - 13x - 39 + 2012 &= 0 \\ 5x^2 + 17x + 2018 &= 0 \end{aligned}$$

Therefore the answer is  $\boxed{5x^2 + 17x + 2018 = 0}$ . The fact that the roots of both these polynomials are complex is irrelevant.

## TEAM ROUND

- Note that  $xy = (x - y)^2$  and  $x \# y = (x + y)^2$ . Then, the numerator is  $(3 - 2)^2(2 + (-1))^2 = 1 \cdot 1 = (1 - 1)^2 = 0$ . We check that the denominator is nonzero (it is 16), so our answer is  $\boxed{0}$ .
- With a distance of  $D$  miles, we can set the traveling time of each trip calculated separately and the overall trip equal:

THE UNIVERSITY OF CHICAGO

PHYSICS DEPARTMENT

PHYSICS 311

LECTURE 1

MECHANICS

LECTURE 2

MECHANICS

LECTURE 3

MECHANICS

LECTURE 4

MECHANICS





$$\frac{D}{42} + \frac{D}{x} = \frac{2D}{48}$$

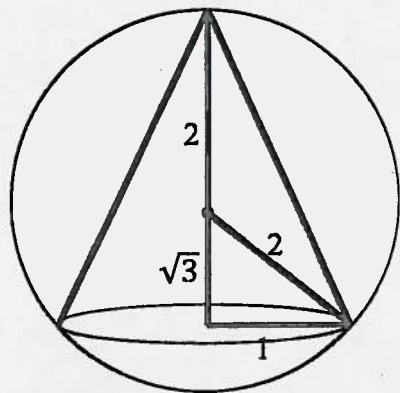
Notice that the quantity  $D$  cancels out, so the total distance is irrelevant! (This is intuitively true.) Therefore, we have

$$\frac{1}{42} + \frac{1}{x} = \frac{1}{24}$$

and it follows that  $x = \boxed{56}$ .

3. We have  $A' = \{5, 6, 7, 8, 9\}$  and  $B' = \{1, 3, 5, 7, 9\}$  and  $A' \cup B' = \{1, 3, 5, 6, 7, 8, 9\}$ . Also,  $A \cup B = \{1, 2, 3, 4, 6, 8\}$ . The intersection of these two sets is  $\{1, 3, 6, 8\}$ . The union with  $A$  is  $\{1, 2, 3, 4, 6, 8\}$ , so the complement is  $\boxed{\{5, 7, 9\}}$ .

4. Use the 30-60-90 right triangle to find lengths:



The volume of the cone is  $\frac{1}{3}\pi(1)^2(2 + \sqrt{3})$  and the volume of the sphere is  $\frac{4}{3}\pi(2)^3$ .

Hence the ratio is  $\boxed{\frac{2 + \sqrt{3}}{32}}$ .

5. Make the substitution  $y = \sqrt{x+1}$ . The equation then becomes

$$6y^3 - 11y^2 + 3y = 0.$$

This factors as  $y(2y-3)(3y-1) = 0$ , so  $y = 0, 3/2, 1/3$ . Recalling that  $y = \sqrt{x+1}$ , we have that  $\boxed{x = -1, 5/4, -8/9}$ .

.



# WORCESTER COUNTY MATHEMATICS LEAGUE



6. We have that  $0.\overline{36} = \frac{36 - 3}{90}$  and  $0.\overline{15} = \frac{15}{99}$ , so the product is  $\frac{33}{90} \cdot \frac{15}{99} = \boxed{\frac{1}{18}}$ .
7. If a quadrilateral can be inscribed in a circle, it is called a *cyclic quadrilateral* and has the property that opposite angles are supplementary (together, opposite angles intercept the entire circle). Therefore,  $180^\circ = m\angle A + m\angle C = (9x + 72)^\circ$  so  $x = 12$ . We then have  $m\angle B = 125^\circ$  so  $m\angle D = (180 - 125)^\circ = \boxed{55^\circ}$ .
8. Christina's speed is  $\frac{3}{4}$  of Angela's. Moira's speed is  $\frac{2}{3}$  of  $\frac{3}{4}$  of Angela's, or  $(\frac{2}{3})(\frac{3}{4}) = \frac{1}{2}$  of it. The average of Christina's and Moira's speeds is therefore  $\frac{5}{8}$  of Angela's, so Angela's speed was  $\boxed{\frac{8}{5}}$  of the average of the other two.
9. We note that  $22 \equiv 1 \pmod{7}$  and  $0 \pmod{11}$  while  $56 \equiv 0 \pmod{7}$  and  $1 \pmod{11}$ . Therefore, integers that are simultaneously  $4 \pmod{7}$  and  $2 \pmod{11}$  are  $4 \cdot 22 + 2 \cdot 56 \pmod{77}$ . Since  $4 \cdot 22 + 2 \cdot 56 = 200$ , all solutions are equivalent to  $200 \pmod{77}$ . The smallest positive solution is therefore  $200 - 2(77) = \boxed{46}$ . [For more information, read about the CHINESE REMAINDER THEOREM]

